Stat 155 Lecture 5 Notes

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1 Solving Two-player Zero-sum Games

1.1 Saddle points

Consider a zero-sum game with the matrix

$$\begin{pmatrix} -1 & 1 & 5 \\ 5 & 3 & 4 \\ 6 & 2 & 1 \end{pmatrix}.$$

Suppose both players choose their 2nd move; the payoff is $a_{2,2} = 3$. Should either player change their strategy? No. This would decrease the payoff for either player. This is called a saddle point, or a pure Nash equilibrium.

Definition 1.1. A pair $(i^*, j^*) \in \{1, \dots, m\} \times \{1, \dots, n\}$ is a *saddle point* for a payoff matrix $A \in \mathbb{R}^{m \times n}$ if

$$\max_i a_{i,j^*} = a_{i^*,j^*} = \min_j a_{i^*,j}.$$

If Player 1 plays i^*, and Player 2 plays j^*, neither player has an incentive to change. Think of saddle points as locally optimal strategies for both players. We will also see that these are globally optimal.

Theorem 1.1. *If (i^*, j^*) is a saddle point for a payoff matrix $A \in \mathbb{R}^{m \times n}$, then*

1. *e_{i^*} is an optimal strategy for Player 1.*
2. *e_{j^*} is an optimal strategy for Player 2.*
3. *The value of the game is a_{i^*,j^*}.*

Proof. We have seen that we should always prefer to play last, but with a saddle point, the opposite inequality is also true:

$$\min_{y \in \Delta_n} \max_{x \in \Delta_m} x^\top A y \geq \max_{x \in \Delta_m} \min_{y \in \Delta_n} x^\top A y$$

$$
\begin{aligned}
&\geq \min_{y\in\Delta_n} e_{i^*}^\top Ay \\
&= e_{i^*}^\top A e_{j^*} \\
&= \max_{x\in\Delta_m} x^\top A e_{j^*} \\
&\geq \min_{y\in\Delta_n} \max_{x\in\Delta_m} x^\top Ay.
\end{aligned}
$$

Observe that $a_{i^*,j^*} = e_{i^*}^\top A e_{j^*}$. $\square$

1.2 Removing dominated pure strategies

Another way to simplify a two-player zero-sum game is by removing dominated rows or columns.

Example 1.1. Here is a game called Plus One. Each player picks a number in $\{1, 2, \ldots, n\}$. If $i = j$, the payoff is 0 If $|i - j| = 1$, the higher number wins 1. If $|i - j| \geq 2$, the higher number loses 2. Here is the payoff matrix.

	1	2	3	4	5	6	$\cdots$	$n-1$	n
1	0	-1	2	2	2	2	$\cdots$	2	2
2	1	0	-1	2	2	2	$\cdots$	2	2
3	-2	1	0	-1	2	2	$\cdots$	2	2
4	-2	-2	1	0	-1	2	$\cdots$	2	2
5	-2	-2	-2	1	0	-1	$\cdots$	2	2
6	-2	-2	-2	-2	1	0	$\cdots$	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	
$n-1$	-2	-2	-2	-2	-2	-2	$\ddots$	0	-1
n	-2	-2	-2	-2	-2	-2	$\cdots$	1	0

If one row is less than another (entry by entry), we can remove the lesser row from the matrix because Player 1 would never choose a strategy in that row. Similarly, we can drop columns that are larger in every entry than other columns. After we remove rows and columns, we get

$$
\begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \end{pmatrix}.
$$

Example 1.2. Here is a game called Miss-by-one. Player 1 and 2 choose numbers $i, j \in \{1, 2, \ldots, 5\}$. Player 1 wins 1 if $|i - j| = 1$; otherwise, the payoff is 0. The matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

If we remove useless rows (1st and 5th) and columns (3rd), we get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

1.3 2×2 games

Consider a zero-sum game with matrix

$$\begin{array}{c|cc} & L & R \\ \hline T & c & d \\ B & a & b \end{array}$$

Assume all the values are different. Without loss of generality, a is the largest. There are six cases, then. The following four cases have saddle points:

1. $a > b > c > d$
2. $a > b > d > c$
3. $a > c > b > d$
4. $a > c > d > b$.

If there are no saddle points, we should equalize mixed strategies. Writing $x_1 = P(T)$, we get

$$V = b + x_1(d - b),$$

$$V = a + x_1(c - a).$$

Solving this gives us

$$x_1 = \frac{a - b}{a - b + d - c}.$$

In more general notation, we get

$$x_1 a_{1,1} + (1 - x_1)a_{2,1} = x_1 a_{1,2} + (1 - x_1)a_{2,2},$$

$$y_1 a_{1,1} + (1 - y_1)a_{1,2} = y_1 a_{2,1} + (1 - y_1)a_{2,2}.$$

Solving gives us

$$x_1 = \frac{a_{2,1} - a_{2,2}}{a_{2,1} - a_{2,2} + a_{1,2} - a_{1,1}},$$

$$y_1 = \frac{a_{1,2} - a_{2,2}}{a_{1,2} - a_{2,2} + a_{2,1} - a_{1,1}}.$$